P1

Tutorial in Week 1

1 Prove that $-(-v) = v$ for every $v \in V$.

Proof

Method 1 : By Proposition 6 on Page 8 of Lecture Note 1, we have

$$-(-v) = (-1)(-v) = (-1)((-1)v) \overset{VS6}{=} ((-1)(-1))v = 1v = v$$

Method 2 : By proposition 2, the additive inverse is unique.

Hence, by definition, $-(-v)$ is the additive inverse of $-v$.

While as $(-v) + v = \vec{0}$, thus v is also the additive inverse of $-v$.

So by uniqueness, $-(-v) = v$.

#

P2

4 The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in 1.19. Which one?

Answer: The empty set ϕ is NOT a vector space. And only the 3rd property (VS3) fails.

Reason: Firstly, (VS3) is stated about the existence of a vector. But for an empty set which contains nothing, the existence obviously fails.

While for other requirements, they are all stated as "$\underbrace{\text{if} \cdots \text{hold}}_{\text{Condition}}$, $\underbrace{\text{then} \cdots \text{hold}}_{\text{Conclusion}}$."

For the empty set, the condition of these statements fail, so we don't need to care about whether their conclusions hold or not.

Note: (VS1), (VS2) or (VS4)—(VS7) fails if we show "if ⋯ hold, then ⋯ NOT hold".

#

13. Let V denote the set of ordered pairs of real numbers. If (a_1, a_2) and (b_1, b_2) are elements of V and $c \in R$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2) \quad \text{and} \quad c(a_1, a_2) = (ca_1, a_2).$$

Is V a vector space over R with these operations? Justify your answer.

Answer: V is NOT a vector space.

Reason: We check (VS1) — (VS7) one by one.

$\boxed{\text{(VS 1)}}$, $\boxed{\text{(VS 2)}}$ are easy for you.

And $\boxed{\text{(VS 3)}}$ holds $\Longleftrightarrow$ $\exists\ \vec{0} = (x_1, x_2) \in V$, such that $\forall\ (a_1, a_2) \in V$, there holds

$(a_1, a_2) + (x_1, x_2) = (a_1, a_2)$.

By the definition of Addition in V, we have

$$(a_1 + x_1,\ a_2 x_2) = (a_1, a_2)$$

$$\Leftrightarrow \begin{cases} a_1 + x_1 = a_1 \\ a_2 x_2 = a_2 \end{cases}, \quad \forall\ a_1, a_2 \in \mathbb{R}.$$

$$\Leftrightarrow \quad x_1 = 0, \quad x_2 = 1. \quad (\text{Take } a_2 = 1)$$

So the unique zero vector in V is $\vec{0} = (0, 1)$.

$\boxed{\text{VS4}}$ holds $\Leftrightarrow$ $\forall\ (a_1, a_2) \in V$, $\exists\ (b_1, b_2) \in V$, such that $(a_1, a_2) + (b_1, b_2) = \vec{0} = (0, 1)$

$$\Rightarrow (a_1 + b_1,\ a_2 b_2) = (0, 1)$$

$$\Rightarrow \begin{cases} a_1 + b_1 = 0 \\ a_2 b_2 = 1 \end{cases}$$

But when $a_2 = 0$, there there doesn't exist $b_2 \in \mathbb{R}$, s.t. $a_2 b_2 = 1$. Thus (VS4) fails.

#

19. Let $V = \{(a_1, a_2) \colon a_1, a_2 \in R\}$. Define addition of elements of V coordinatewise, and for (a_1, a_2) in V and $c \in R$, define

$$c(a_1, a_2) = \begin{cases} (0, 0) & \text{if } c = 0 \\ \left(ca_1, \dfrac{a_2}{c}\right) & \text{if } c \neq 0. \end{cases}$$

Is V a vector space over R with these operations? Justify your answer.

Answer: V is NOT a vector space.

Proof: (VS 1) – (VS 4) are easy since the addition are the same to the one in real space $\mathbb{R}$.

(VS 5) is also obvious by the definition of scalar multiplication.

P4

Check $\boxed{(VS6)}$: $\forall\ (a_1,a_2)\in V,\ \forall\ c_1,c_2\in\mathbb{R},$

$$(c_1c_2)(a_1,a_2) \overset{?}{=} c_1(c_2(a_1,a_2)).$$

Case 1: If $c_1=0$, then $c_1c_2=0$.

So by def., $\begin{cases}(c_1c_2)(a_1,a_2)=(0,0)\\ c_1(c_2(a_1,a_2))=(0,0)\end{cases}$ ✓

Case 2: If $c_2=0$, then $c_1c_2=0$

So $c_1(c_2(a_1,a_2)) = c_1(0,0) = (0,0)$

$= (c_1c_2)(a_1,a_2)$. ✓

Case 3: If $c_1\neq 0$ and $c_2\neq 0$, then $c_1c_2\neq 0$.

So $(c_1c_2)(a_1,a_2) = \left(c_1c_2a_1, \dfrac{a_2}{c_1c_2}\right)$ ✓

And $c_1(c_2(a_1,a_2)) = c_1\left(c_2a_1, \dfrac{a_2}{c_2}\right) = \left(c_1c_2a_1, \dfrac{a_2}{c_1c_2}\right)$

So (VS6) is true.

Check $\boxed{(VS7)}$: (i). $\forall\ c\in\mathbb{R}, \forall\ u=(a_1,a_2),\ v=(b_1,b_2)\in V,$

$$c(u+v) \overset{?}{=} cu+cv$$

(ii). $\forall\ c_1,c_2\in\mathbb{R},\ \forall\ u=(a_1,a_2)\in V,$

$$(c_1+c_2)u \overset{?}{=} c_1u+c_2u.$$

Check (i): Case 1. if $c=0$, then $c(u+v)=(0,0)$ ✓

$cu+cv=(0,0)+(0,0)=(0,0)$

Case 2. if $c\neq 0$, then $c(u+v)=c(a_1+b_1, a_2+b_2)$

$= \left(c(a_1+b_1), \dfrac{a_2+b_2}{c}\right)$

And $cu+cv = \left(ca_1, \dfrac{a_2}{c}\right) + \left(cb_1, \dfrac{b_2}{c}\right)$

$= \left(c(a_1+b_1), \dfrac{a_2+b_2}{c}\right)$. ✓

So (VS7).(i) is true

Check (ii): We need to care $c_1+c_2 \overset{?}{=} 0$, $c_1 \overset{?}{=} 0$, $c_2 \overset{?}{=} 0$.

Case 1: If $c_1 = c_2 = 0$, then $c_1 + c_2 = 0$, so we have
$(c_1 + c_2) u = 0 (a_1, a_2) = (0, 0)$
And $c_1 u + c_2 u = 0u + 0u = (0,0) + (0,0) = (0,0)$. ✓

Case 2: If $c_1 \neq 0$ and $c_1 + c_2 = 0$, then $c_2 = -c_1 \neq 0$.
So $(c_1 + c_2) u = (0, 0)$
And $c_1 u + c_2 u = \left(c_1 a_1, \frac{a_2}{c_1} \right) + \left(c_2 a_1, \frac{a_2}{c_2} \right)$
$= \left((c_1 + c_2) a_1, \frac{c_1 + c_2}{c_1 c_2} a_2 \right) = (0, 0)$ ✓

Case 3: If $c_2 \neq 0$ and $c_1 + c_2 = 0$, which is same with Case 2.

Case 4: If $c_1 + c_2 \neq 0$, $c_1 \neq 0$ and $c_2 \neq 0$, then
$(c_1 + c_2) u = \left((c_1 + c_2) a_1, \frac{a_2}{c_1 + c_2} \right)$
And $c_1 u + c_2 u = \left(c_1 a_1, \frac{a_2}{c_1} \right) + \left(c_2 a_1, \frac{a_2}{c_2} \right)$
$= \left((c_1 + c_2) a_1, \frac{c_1 + c_2}{c_1 c_2} a_2 \right)$

But if $a_2 \neq 0$ and $\frac{c_1 + c_2}{c_1 c_2} \neq \frac{1}{c_1 + c_2}$ true in general.

$(c_1 + c_2) u \neq c_1 u + c_2 u$,
so (VS 7).(ii) fails. #

Review Gaussian Elimination

How to solve the system of linear equations

P6

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

or written in the matrix form: $Ax=b$.

where $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \in \mathbb{R}^{n\times n}$

$b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in \mathbb{R}^n$, $\quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ (n unknown variables)

The problem has three possibilities

(1). there is NO solution.

(2). there is a unique solution.

(3). there are infinitely many solutions.

Gaussian Elimination Method is using the row operations on the augmented matrix

$$[A \mid b] = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & b_n \end{array}\right]$$, in order to achieve

an upper triangular matrix, which is

$$\left[\begin{array}{cccc|c} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} & \tilde{b}_1 \\ 0 & \tilde{a}_{22} & \cdots & \tilde{a}_{2n} & \tilde{b}_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \tilde{a}_{nn} & \tilde{b}_n \end{array}\right]$$

Correspondingly, the system of equations becomes

$$\begin{cases} \tilde{a}_{11} x_1 + \tilde{a}_{12} x_2 + \cdots + \tilde{a}_{1n} x_n = \tilde{b}_1 \\ \qquad\quad \tilde{a}_{22} x_2 + \cdots + \tilde{a}_{2n} x_n = \tilde{b}_2 \\ \qquad\qquad \vdots \\ \qquad\qquad\qquad\qquad \tilde{a}_{nn} x_n = \tilde{b}_n \end{cases}$$

Then we can use "back substitution" to solve x_n, $x_{n-1}, \cdots, x_1$ one by one, if $\tilde{a}_{nn}, \cdots, \tilde{a}_{22}, \tilde{a}_{11}$ are all not zero.

If some of diagonal entries $\tilde{a}_{11}, \tilde{a}_{22}, \cdots, \tilde{a}_{nn}$ are zero, then there may exist infinitely many solutions or there may be no solution.

<u>Example</u>: Solve $\begin{cases} x_1 - x_3 = 1 \\ x_1 + x_2 + 2x_3 = 3 \\ 3x_1 + x_2 = 5 \end{cases}$

<u>Sol</u>. Its augmented matrix $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 1 & 1 & 2 & 3 \\ 3 & 1 & 0 & 5 \end{array}\right]$

$$\xrightarrow[\text{row}_1 \times (-3) + \text{row}_3]{\text{row}_1 \times (-1) + \text{row}_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & 3 & 2 \end{array}\right]$$

$$\xrightarrow{\text{row}_2 \times (-1) + \text{row}_3} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Correspondingly, we have $\begin{cases} x_1 - x_3 = 1 \\ x_2 + 3x_3 = 2 \end{cases}$

$\Rightarrow \begin{cases} x_1 = 1 + x_3 \\ x_2 = 2 - 3x_3 \end{cases}$

So $\forall\ x_3 \in \mathbb{R}$, $x = (1 + x_3,\ 2 - 3x_3,\ x_3)$ is a solution.

That is, there exist infinitely many solutions.

#